relatively well with the final shape of the streamlines. Details of the stability analysis and other aspects of the numerical scheme can be found in Ref. 5.

Results

For the present calculations, grids having 32×8 mesh cells were used. The convergence rates for both planar and axisymmetric cases were 0.92-0.95 per work unit, where one work unit corresponds to the computational time required for a single ADI sweep on the finest grid. The global features of the solution typically have converged within approximately 30 work units; such a computation requires 4.5 s of CPU time on the IBM 370/168 computer.

Results for the transonic flow through a planar nozzle are presented in Fig. 1. For this case, the nozzle throat radius is 1.82 and the boundary conditions are such that a shock wave spans the entire nozzle. Mach number contours for this case are plotted and are compared with those of Emmons[1] for the same case. Emmons used a primitive form of relaxation method that is essentially equivalent to the present method for subsonic flows. However, Emmons used a trial-and-error method for fitting the shock instead of capturing it. He also included the effects of the entropy jump and the vorticity field generated by the shock, which is the primary difference from the present method. To compare the two results, the shock position at the wall is matched by adjusting the potential value on the downstream boundary in the present calculation. There is some discrepancy between the two solutions downstream of the shock, which almost certainly can be attributed to the effect of the entropy jump included in Emmons' calculation.

In Fig. 2, results for the fully expanded flow through an axisymmetric nozzle are compared with those of Dutton and Addy.[6] They solved for a circular arc nozzle geometry with unit radius of curvature, and both the experimental and computed results are presented. Since Dutton and Addy used a perturbation method of third order, their results are independent of the precise wall shape near the throat. To minimize the effects of the geometric differences, a short domain was chosen for the present calculation; the termination of the contours for the present calculation near the downstream boundary reflects the location of the boundary of the computational domain. The basic features of the flow are essentially the same for both methods, with particularly good agreement in the throat region. As might be expected, better agreement is obtained throughout the supersonic region when using the second-order accurate form of the viscosity.

Finally, a comparison between the present work and an analytical solution for a flow containing a shock is presented. Lin and Shen[7] have obtained analytical solutions for two-dimensional and axisymmetric transonic flows in slender hyperbolic nozzles including shock waves. Their matched asymptotic analysis was carried out using expansions for both the velocity potential and stream function formulations of the equations of motion. Because the expansions used in these calculations were necessarily truncated, the results from the two formulations are slightly different. In Fig. 3, the velocity distributions along the wall and the axis of symmetry are shown for the case of a radius of curvature at the throat equal to 10. It is interesting to note that the present results match more closely with the analytical solutions for the stream function formulation than for the velocity potential formulation. This may be explained by the fact that the stream function formulation keeps the mass flow exactly constant, as does the present formulation. The shock shapes for both formulations are in good agreement with the results of the present calculations.

Conclusion

Transonic flows in hyperbolic nozzles have been calculated using a finite volume, transonic potential method. A multigrid ADI method was used to solve the resulting difference equations. The scheme converges rapidly, and results are in good agreement with those of earlier workers, including analytical results for flows containing shock waves.

Acknowledgment

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Vorticity with Variable Viscosity

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THE formulations developed in Ref. 1 are eminently useful in the analysis of vorticity mechanics in complex flows. As noted therein, the focus of the study was not viscous compressible flow. However, some of the equations retain the viscosity inside various derivatives, implying that all the variable viscosity effects are accounted for, at least formally. The purpose of this Note is to document some effects of the additional terms required to complete the formulation when variable viscosity flow is the subject of interest.

Following the notation in Ref. 2, the equations of motion may be written in the following form:

$$\rho \mathrm{D}u_i/\mathrm{D}t = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{2\mu\left(e_{ij} - \frac{1}{3}\Delta\delta_{ij}\right)\right\} \tag{1}$$

where e_{ij} is the symmetric part of the deformation

$$e_{ij} \equiv \frac{(\partial u_i/\partial x_j + \partial u_j/\partial x_i)}{2} \tag{2}$$

$\Delta \equiv e_{ij}$ and δ_{ij} is the Kronecker delta. Density, vorticity, velocity, normal stress, body force, and the coefficient of viscosity are assigned their traditional nomenclature.

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Straightforward differentiation of the viscous terms in Eq. (1) allows them to be written

$$\mu\left(\nabla^2 u_i + \frac{\partial\Delta/\partial x_i}{3}\right) + \frac{2(e_{ij}\partial\mu/\partial x_j - \Delta\partial\mu/\partial x_i)}{3} \tag{3}$$

In addition, the appropriate vector calculus identity allows this expression to be rewritten as

$$-\mu(\nabla\times\omega)_i + \frac{4}{3}\frac{\mu\partial\Delta}{\partial x_i} + 2\left(e_{ij}\frac{\partial\mu}{\partial x_j} - \frac{\Delta\partial\mu}{\partial x_i}\right) \tag{4}$$

In order to derive the vorticity transport equation, this expression is substituted into the equations of motion, each term is divided by density ρ, and the entire expression is subjected to the curl operator. Comparing this with Eq. (1) in Ref. 1, it is clear that the additional variable viscosity terms

$$2\nabla\times\frac{(e_{ij}\partial\mu/\partial x_j - \Delta\partial\mu/\partial x_i)}{\rho} \tag{5}$$

must be addressed for general viscous flows.

The focus of interest in this study is the effect of the first term in flows for which the approximation $\rho\simeq$constant, $\Delta\simeq 0$ is a good one, yet μ must be allowed to vary. In addition to various laminar flows, this situation is applicable to turbulence modeling as well, in which μ is related to the local flowfield properties. We consider, therefore, the following term in the general vorticity transport balance:

$$\left(\frac{2}{\rho}\right)\nabla\times\left(e_{ij}\frac{\partial\mu}{\partial x_j}\right) \tag{6}$$

For cases of interest in which μ varies predominantly in one direction, herein x_2, the components of the curl in the three orthogonal directions are

$$\frac{\partial}{\partial x_2}\left(\frac{e_{32}\partial\mu}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{e_{22}\partial\mu}{\partial x_2}\right); \quad x_1$$

$$\frac{\partial}{\partial x_1}\left(\frac{e_{32}\partial\mu}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{e_{12}\partial\mu}{\partial x_2}\right); \quad x_2$$

$$\frac{\partial}{\partial x_1}\left(\frac{e_{22}\partial\mu}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{e_{12}\partial\mu}{\partial x_2}\right); \quad x_3 \tag{7}$$

If x_1, x_2, and x_3 are associated with the streamwise, normal, and spanwise coordinates of a shear flow, it is clear that explicit three-dimensionality is required to impact streamwise vorticity from variable viscosity terms. However, in two-dimensional flow, the spanwise vorticity is affected by the component of the curl in x_3:

$$\frac{\partial\mu}{\partial x_2}\left(\frac{\partial e_{22}}{\partial x_1} - \frac{\partial e_{12}}{\partial x_2}\right) - \frac{\partial^2\mu}{\partial x_2^2}e_{12} \tag{8}$$

which may be written

$$\frac{\partial\mu}{\partial x_2}\left(\frac{\partial}{\partial x_2}\frac{\omega_3}{2}\right) - \frac{\partial^2\mu}{\partial x_2^2}e_{12} \tag{9}$$

If, in addition, $u_2 \ll u_1$, this expression further reduces to one-half of

$$\frac{\partial\mu}{\partial x_2}\left(\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2}\right) - \frac{\partial^2\mu}{\partial x_2^2}\frac{\partial u_1}{\partial x_2} \tag{10}$$

or

$$\frac{\partial\mu}{\partial x_2}\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2\mu}{\partial x_2^2}\frac{\partial u_1}{\partial x_2} \tag{11}$$

If, for example, μ exhibits some empirical dependence on, say, $\partial u_1/\partial x_2$, the vorticity generation will respond to a term proportional to

$$\mu\frac{\partial^2\mu}{\partial x_2^2} - \left(\frac{\partial\mu}{\partial x_2}\right)^2 \tag{12}$$

which, clearly, can undergo several sign changes across a shear flow and contribute positive or negative terms to the vorticity balance. The accurate representation of variable viscosity effects is, therefore, critical in representing the detailed vortical mechanics.

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Triple-Velocity Products in a Channel with a Backward-Facing Step

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Introduction

PREDICTING physical behavior of separating and recirculating flows is an important aspect in aeronautics and in many industrial problems. To improve turbulence modeling for such complex turbulent flows, it is necessary to conduct an extensive study of turbulence behavior in reattaching shear flows.

Experimental observations by Chandrsuda and Bradshaw[1] show that when the separated flow reattaches on a solid wall, the separated mixing layer begins to change rapidly, and large-scale eddies are suppressed due to the solid wall, leading to a marked decrease in the triple-velocity products of turbulence fluctuating velocities toward the solid surface from the region of maximum turbulence intensity.

This paper focuses on the evaluation of the triple-velocity products in the reattaching and redeveloping region behind a step. As discussed above, the change in triple-velocity products is significant in the wake region, resulting in a considerable variation in the diffusion rate of the Reynolds stresses. Thus, it is important to reevaluate the existing models of the third-order closure for better understanding of such diffusion processes in the separated shear layers. In the present paper four models of the third-order closure are examined, and the results are compared with the experimental data of Chandrsuda and Bradshaw.[1] The models considered are those proposed by Daly and Harlow,[2] Hanjalic and Launder,[3] Shir,[4] and Cormack et al.[5]

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